

SPECTRAL FUNCTIONS RELATED TO SOME FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we consider fractional higher-order stochastic differential equations of the form

$$\left(\mu + c_\alpha \frac{d^\alpha}{d(-t)^\alpha} \right)^\beta X(t) = \mathcal{E}(t), \quad t \geq 0, \mu > 0, \beta > 0, \alpha \in (0, 1) \cup \mathbb{N}$$

where $\mathcal{E}(t)$ is a Gaussian white noise. We derive stochastic processes satisfying the above equations of which we obtain explicitly the covariance functions and the spectral functions.

1. INTRODUCTION

In this paper we consider fractional stochastic ordinary differential equations of different form where the stochastic component is represented by a Gaussian white noise. Most of the fractional equations considered here are related to the higher-order heat equations and thus are connected with pseudoprocesses.

The first part of the paper considers the following stochastic differential equation

$$\left(\mu + \frac{d^\alpha}{d(-t)^\alpha} \right)^\beta X(t) = \mathcal{E}(t), \quad \beta > 0, 0 < \alpha < 1, \mu > 0 \quad (1.1)$$

where $\frac{d^\alpha}{d(-t)^\alpha}$ represents the upper-Weyl fractional derivative. We obtain a representation of the solution to (1.1) in the form

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty dz \int_0^\infty ds s^{\beta-1} e^{-s\mu} h_\alpha(z, s) \mathcal{E}(t+z) \quad (1.2)$$

where $h_\alpha(z, s)$, $z, s \geq 0$, is the density function of a positively skewed stable process $H_\alpha(t)$, $t \geq 0$ of order $\alpha \in (0, 1)$, that is with Laplace transform

$$\int_0^\infty e^{-\xi z} h_\alpha(z, s) dz = e^{-s\xi^\alpha}, \quad \xi \geq 0.$$

For (1.2), we obtain the spectral function

$$f(\tau) = \frac{\sigma^2}{(\mu^2 + 2|\tau|^\alpha \mu \cos \frac{\pi\alpha}{2} + |\tau|^{2\alpha})^\beta}, \quad \tau \in \mathbb{R} \quad (1.3)$$

and the related covariance function.

The second type of stochastic differential equations we consider has the form

$$\left(\mu + (-1)^n \frac{\partial^{2n}}{\partial t^{2n}} \right)^\beta X(t) = \mathcal{E}(t), \quad \beta > 0, \mu > 0, n \geq 1, \quad (1.4)$$

Date: July 8, 2015.

2000 Mathematics Subject Classification. 60K99; 60G60.

Key words and phrases. Higher-order heat equations, Weyl fractional derivatives, Airy functions, spectral functions.

where $\mathcal{E}(t)$ is a Gaussian white noise. The representation of the solution to (1.4) is

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty dw w^{\beta-1} e^{-\mu w} \int_{-\infty}^{+\infty} dx u_{2n}(x, w) \mathcal{E}(t+x) \quad (1.5)$$

where $u_{2n}(x, w)$, $x \in \mathbb{R}$, $w \geq 0$ is the fundamental solution to $2n$ -th order heat equation

$$\frac{\partial u}{\partial w}(x, w) = (-1)^{n+1} \frac{\partial^{2n} u}{\partial x^{2n}}(x, w) \quad (1.6)$$

The autocovariance function of the process (1.5) can be written as

$$\mathbb{E}X(t)X(t+h) = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^\infty dw w^{2\beta-1} e^{-\mu w} u_{2n}(h, w) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E}u_{2n}(h, W_{2\beta}) \quad (1.7)$$

where $W_{2\beta}$ is a gamma r.v. with parameters μ and 2β . The spectral function $f(\tau)$ associated with (1.7) has the fine form

$$f(\tau) = \frac{\sigma^2}{(\mu + \tau^{2n})^{2\beta}} \quad (1.8)$$

For $n = 1$, (1.6) is the classical heat equation, $u_2(x, w) = \frac{e^{-\frac{x^2}{4w}}}{\sqrt{4\pi w}}$ and, from (1.7) we obtain an explicit form of the covariance function in terms of the modified Bessel functions. In connection with the equations of the form (1.6) the so-called pseudoprocesses, first introduced at the beginning of the Sixties ([6]) have been constructed. The solutions to (1.6) are sign-varying and their structure has been explored by means of the steepest descent method ([9; 1]) and their representation has been recently given by [10].

For the fractional odd-order stochastic differential equation

$$\left(\mu + (-1)^n \frac{d^{2n+1}}{dt^{2n+1}} \right)^\beta X(t) = \mathcal{E}(t), \quad n = 1, 2, \dots \quad (1.9)$$

the solution has the structure

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty dw w^{\beta-1} \int_{\mathbb{R}} dx u_{2n+1}(x, w) \mathcal{E}(t+x) \quad (1.10)$$

where $u_{2n+1}(x, w)$, $x \in \mathbb{R}$, $w \geq 0$ is the fundamental solution to

$$\frac{\partial u}{\partial w}(x, w) = (-1)^n \frac{\partial^{2n+1} u}{\partial x^{2n+1}}(x, w). \quad (1.11)$$

The solutions u_{2n+1} and u_{2n} are substantially different in their behaviour and structure as shown in [10] and [7].

A special attention has been devoted to the case $n = 1$ for which (1.10) takes the interesting form

$$X_3(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty dw w^{\beta-1} \int_{\mathbb{R}} dx \frac{1}{\sqrt[3]{3w}} Ai\left(\frac{x}{\sqrt[3]{3w}}\right) \mathcal{E}(t+x) \quad (1.12)$$

where $Ai(\cdot)$ is the first-type Airy function. The process X_3 can also be represented as

$$X_3(t) = \frac{1}{\mu^\beta} \mathbb{E} \mathcal{E}(t + Y_3(W_\beta)) \quad (1.13)$$

where the mean value must be meant w.r.t. $Y_3(W_\beta)$ and Y_3 is the pseudoprocess related to equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} \quad (1.14)$$

and W_β is a Gamma-distributed r.v. with parameters β, μ independent from Y_3 . The autocovariance function of X_3 has the following form

$$\mathbb{E}X_3(t)X_3(t+h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\frac{1}{\sqrt[3]{3W_{2\beta}}} \text{Ai} \left(\frac{h}{\sqrt[3]{3W_{2\beta}}} \right) \right] \quad (1.15)$$

where $W_{2\beta}$ is the sum of two independent r.v.'s W_β . For the solution to the general odd-order stochastic equation we obtain the covariance function

$$\mathbb{E}X(t)X(t+h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} [u_{2n+1}(h, W_{2\beta})] \quad (1.16)$$

Of course, the Fourier transform of (1.16) becomes

$$f(\tau) = \frac{1}{\mu^{2\beta}} \int_{\mathbb{R}} e^{i\tau h} \sigma^2 \mathbb{E} [u_{2n+1}(h, W_{2\beta})] dh = \left(\frac{\mu}{\mu + i\tau^{2n+1}} \right)^{2\beta}. \quad (1.17)$$

Stochastic fractional differential equations similar to those dealt with here have been analysed in [2], [3] and [5]. In our paper we consider equations where different operators are involved.

2. A STOCHASTIC EQUATION INVOLVING FRACTIONAL POWERS OF FRACTIONAL OPERATORS

In this section we consider the following generalization of the Gay and Heyde equation (see [3])

$$\left(\mu + \frac{d^\alpha}{d(-t)^\alpha} \right)^\beta X(t) = \mathcal{E}(t), \quad \beta > 0, 0 < \alpha < 1, \mu > 0 \quad (2.1)$$

where $\mathcal{E}(t)$, $t > 0$, is a Gaussian white noise with

$$\mathbb{E}\mathcal{E}(t)\mathcal{E}(s) = \begin{cases} \sigma^2, & t = s \\ 0, & t \neq s \end{cases}. \quad (2.2)$$

The fractional derivative appearing in (2.1) must be meant, for $0 < \alpha \leq 1$, as

$$\frac{d^\alpha}{d(-t)^\alpha} f(t) = - \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty \frac{f(s)}{(s-t)^\alpha} ds \quad (2.3)$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(t) - f(t+w)}{w^{\alpha+1}} dw. \quad (2.4)$$

For information on fractional derivatives of this form, called also Marchaud derivatives, consult [11, pag. 111]. For $\lambda \geq 0$, we introduce the Laplace transform

$$\mathcal{L} \left[\frac{d^\alpha f}{d(-t)^\alpha} \right] (\lambda) = \int_0^\infty e^{\lambda t} \frac{d^\alpha}{d(-t)^\alpha} f(t) dt = \lambda^\alpha \mathcal{L}[f](\lambda) \quad (2.5)$$

which can be immediately obtained by considering that

$$\mathcal{L} \left[\frac{d^\alpha f}{d(-t)^\alpha} \right] (\lambda) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(\mathcal{L}[f](\lambda) - e^{-w\lambda} \mathcal{L}[f](\lambda) \right) \frac{dw}{w^{\alpha+1}} \quad (2.6)$$

for a function f such that $e^{\lambda t} f(t) \in L^1([0, \infty))$.

Theorem 2.1. *The representation of a solution to the equation (2.1) can be written as*

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty dz \int_0^\infty ds s^{\beta-1} e^{-s\mu} h_\alpha(z, s) \mathcal{E}(t+z) \quad (2.7)$$

Proof. The solution to the equation (2.1) can be obtained as follows

$$\begin{aligned}
X(t) &= \left(\frac{d^\alpha}{d(-t)^\alpha} + \mu \right)^{-\beta} \mathcal{E}(t) \\
&= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s\mu - s \frac{d^\alpha}{d(-t)^\alpha}} \mathcal{E}(t) ds \\
&= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s\mu} \left\{ e^{-s \frac{d^\alpha}{d(-t)^\alpha}} \mathcal{E}(t) \right\} ds.
\end{aligned} \tag{2.8}$$

Now, for the stable subordinator $H^\alpha(t)$, $t > 0$, we have that

$$\begin{aligned}
e^{-s \frac{d^\alpha}{d(-t)^\alpha}} \mathcal{E}(t) &= \mathbb{E} e^{H^\alpha(s) \frac{d}{dt}} \mathcal{E}(t) \\
&= \int_0^\infty dz h_\alpha(z, s) e^{z \frac{d}{dt}} \mathcal{E}(t) \\
&= \int_0^\infty dz h_\alpha(z, s) \mathcal{E}(t + z)
\end{aligned} \tag{2.9}$$

where $h_\alpha(z, s)$ is the probability law of $H^\alpha(s)$, $s > 0$. In the last step of (2.9) we used the translation property

$$e^{z \frac{d}{dt}} \mathcal{E}(t) = \mathcal{E}(t + z). \tag{2.10}$$

This is because

$$e^{z \frac{d}{dt}} \phi(t) = \sum_{k=0}^\infty \frac{z^k}{k!} \frac{d^k}{dt^k} \phi(t). \tag{2.11}$$

In view of the Taylor expansion

$$f(x) = \sum_{k=0}^\infty f^{(k)}(x_0) \frac{(x - x_0)^k}{k!} \tag{2.12}$$

with $x_0 = t$ and $x = t + z$ we have that

$$e^{z \frac{d}{dt}} \phi(t) = \phi(t + z) \tag{2.13}$$

which holds for a bounded and continuous function $\phi : [0, \infty) \mapsto [0, \infty)$. Since we can find a sequence of r.v.'s $\{a_j\}_{j \in \mathbb{N}}$ and an orthonormal set, say $\{\phi_j\}_{j \in \mathbb{N}}$, for which (2.13) holds true $\forall j$ and such that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\| \mathcal{E} - \sum_{j=1}^N a_j \phi_j \right\|_2 = 0,$$

we can write (2.10). Therefore,

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty dz \int_0^\infty ds s^{\beta-1} e^{-s\mu} h_\alpha(z, s) \mathcal{E}(t + z) \tag{2.14}$$

is the formal solution to the fractional equation (2.1) with representation, in mean square sense, given by

$$X(t) = \frac{1}{\mu^\beta} \sum_{j \in \mathbb{N}} a_j \mathbb{E}[\phi_j(t + H^\alpha(W_\beta))], \quad t > 0. \tag{2.15}$$

□

Remark 2.1. For the case $\alpha = 1$, $h_\alpha(z, s) = \delta(z - s)$ where δ is the Dirac delta function and from (2.10) we infer that

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\mu s} s^{\beta-1} \mathcal{E}(t + s) ds \tag{2.16}$$

solves the fractional equation

$$\left(\mu - \frac{d}{dt}\right)^\beta X(t) = \mathcal{E}(t). \quad (2.17)$$

Consult on this point [5].

A direct proof is also possible because from (2.8) we have that

$$\begin{aligned} X(t) &= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-\mu s} e^{s \frac{d}{dt}} \mathcal{E}(t) ds \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-\mu s} \mathcal{E}(t+s) ds. \end{aligned} \quad (2.18)$$

In the last step we applied (2.10).

Remark 2.2. For $\alpha = 1$ and $\beta = 1$, we observe that (2.7) becomes the Ornstein-Uhlenbeck process.

Our next step is the evaluation of the Fourier transform of the covariance function of the solution to the differential equation (2.1). Let

$$f(\tau) = \int_{-\infty}^{+\infty} e^{i\tau h} \text{Cov}_X(h) dh$$

where

$$\text{Cov}_X(h) = \mathbb{E}[X(t+h)X(t)]$$

with $\mathbb{E}X(t) = 0$.

Theorem 2.2. The spectral density of (2.7) is

$$f(\tau) = \frac{\sigma^2}{(\mu^2 + 2|\tau|^\alpha \mu \cos \frac{\pi\alpha}{2} + |\tau|^{2\alpha})^\beta}, \quad \tau \in \mathbb{R}, 0 < \alpha < 1, \beta > 0. \quad (2.19)$$

Proof. The Fourier transform of the covariance function of lag $h = t_2 - t_1$ of (2.7) is given by

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{i\tau(t_2-t_1)} \mathbb{E}X(t_1)X(t_2) dt_1 dt_2 \\ &= \frac{1}{\Gamma^2(\beta)} \int_0^\infty \int_0^\infty e^{i\tau(t_2-t_1)} dt_1 dt_2 \int_0^\infty dz_1 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_0^\infty dz_2 s_1^{\beta-1} s_2^{\beta-1} \\ &\quad \times e^{-(s_1+s_2)\mu} h_\alpha(z_1, s_1) h_\alpha(z_2, s_2) \mathbb{E}\mathcal{E}(t_1+z_1)\mathcal{E}(t_2+z_2) \end{aligned}$$

where

$$\mathbb{E}\mathcal{E}(t_1+z_1)\mathcal{E}(t_2+z_2) = \begin{cases} \sigma^2, & h = z_1 - z_2 \\ 0, & h \neq z_1 - z_2 \end{cases}. \quad (2.20)$$

Thus,

$$\begin{aligned} \int_0^\infty \int_0^\infty e^{i\tau(t_2-t_1)} \mathbb{E}X(t_1)X(t_2) dt_1 dt_2 &= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty dz_1 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_0^\infty dz_2 s_1^{\beta-1} s_2^{\beta-1} \\ &\quad \times e^{-(s_1+s_2)\mu} h_\alpha(z_1, s_1) h_\alpha(z_2, s_2) e^{i\tau(z_1-z_2)}. \end{aligned}$$

By considering the characteristic function of a positively-skewed stable process with law h_α , we have that

$$\int_0^\infty e^{i\tau z_1} h_\alpha(z_1, s_1) dz_1 = e^{-(i\tau)^\alpha s_1} = e^{-s_1|\tau|^\alpha e^{-i\frac{\pi}{2} \text{sgn } \tau}}, \quad (2.21)$$

and

$$\int_0^\infty e^{-i\tau z_2} h_\alpha(z_2, s_2) dz_2 = e^{-(i\tau)^\alpha s_2} = e^{-s_2|\tau|^\alpha e^{i\frac{\pi}{2} \text{sgn } \tau}}. \quad (2.22)$$

Thus, we obtain that

$$\begin{aligned}
& \int_0^\infty \int_0^\infty e^{i\tau(t_2-t_1)} \mathbb{E}X(t_1)X(t_2) dt_1 dt_2 \\
&= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty ds_1 \int_0^\infty ds_2 s_1^{\beta-1} s_2^{\beta-1} e^{-(s_1+s_2)\mu} e^{-(i\tau)^\alpha s_2 - (-i\tau)^\alpha s_1} \\
&= \frac{\sigma^2}{\left(\mu + |\tau|^\alpha e^{-\frac{i\pi\alpha}{2} \operatorname{sgn} \tau}\right)^\beta \left(\mu + |\tau|^\alpha e^{\frac{i\pi\alpha}{2} \operatorname{sgn} \tau}\right)^\beta} \\
&= \frac{\sigma^2}{\left(\mu^2 + 2|\tau|^\alpha \mu \cos \frac{\pi\alpha}{2} + |\tau|^{2\alpha}\right)^\beta}.
\end{aligned}$$

□

Remark 2.3. In the special case $\alpha = 1$ the result above simplifies and yields

$$f(\tau) = \frac{\sigma^2}{(\mu^2 + \tau^2)^\beta}. \quad (2.23)$$

We note that for $\beta = 1$, (2.23) becomes the spectral function of the Ornstein-Uhlenbeck process. Processes with the spectral function f are dealt with, for example, in [2] where also space-time random fields governed by stochastic equations are considered. The covariance function is given by

$$\begin{aligned}
\operatorname{Cov}_X(h) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\tau h} f(\tau) d\tau \\
&= \frac{\sigma^2}{2\pi} \int_{\mathbb{R}} e^{-i\tau h} \left(\frac{1}{\Gamma(\beta)} \int_0^\infty z^{\beta-1} e^{-z\mu^2 - z\tau^2} dz \right) d\tau \\
&= \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty z^{\beta-1} e^{-z\mu^2} \left(\frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\tau h - z\tau^2} d\tau \right) dz \\
&= \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty z^{\beta-1} e^{-z\mu^2} \frac{e^{-\frac{h^2}{4z}}}{\sqrt{4\pi z}} dz \\
&= \frac{\sigma^2}{2\Gamma(\beta)\Gamma(\frac{1}{2})} \int_0^\infty z^{\beta-\frac{1}{2}-1} e^{-z\mu^2 - \frac{h^2}{4z}} dz \\
&= \frac{\sigma^2}{\Gamma(\beta)\Gamma(\frac{1}{2})} \left(\frac{|h|}{2\mu} \right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}(\mu|h|), \quad h \geq 0
\end{aligned}$$

where K_ν is the modified Bessel function with integral representation given by

$$\int_0^\infty x^{\nu-1} \exp\{-\beta x^p - \alpha x^{-p}\} dx = \frac{2}{p} \left(\frac{\alpha}{\beta} \right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}} \left(2\sqrt{\alpha\beta} \right), \quad p, \alpha, \beta, \nu > 0 \quad (2.24)$$

(see for example [4], formula 3.478). We observe that $K_\nu = K_{-\nu}$ and $K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$. Moreover,

$$K_\nu(x) \approx \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu} \quad \text{for } x \rightarrow 0^+ \quad (2.25)$$

([8, pag. 136]) and

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \quad \text{for } x \rightarrow \infty. \quad (2.26)$$

Thus, we get that

$$\operatorname{Cov}_X(h) \approx \mu^{1-2\beta}, \quad \text{for } h \rightarrow 0^+ \quad (2.27)$$

and

$$\text{Cov}_X(h) \approx \left(\frac{h}{\mu}\right)^\beta \frac{1}{h} e^{-\mu h}, \quad \text{for } h \rightarrow \infty. \quad (2.28)$$

We study the covariance of (1.2). Recall that, a stable process S of order α with density g is characterized by

$$\widehat{g}(\xi, t) = \mathbb{E} e^{i\xi S(t)} = e^{-\sigma^2 |\xi|^\alpha t}, \quad \alpha \in (0, 2].$$

Consider two independent stable processes $S_1(w)$, $S_2(w)$, $w \geq 0$, with $\sigma_1^2 = 1$ and $\sigma_2^2 = 2\mu \cos \frac{\pi\alpha}{2}$. Let $g_1(x, w)$, $x \in \mathbb{R}$, $w \geq 0$ and $g_2(x, w)$, $x \in \mathbb{R}$, $w \geq 0$ be the corresponding density laws. Then, the following result holds true.

Theorem 2.3. *The covariance function of (1.2) is*

$$\text{Cov}_X(h) = \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-w\mu^2} \int_{-\infty}^{+\infty} g_1(h-z, w) g_2(z, w) dz dw \quad (2.29)$$

or

$$\text{Cov}_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} g_{S_1+S_2}(h, W_\beta) \quad (2.30)$$

and W_β is a gamma r.v. with parameters μ^2, β .

Proof. Notice that

$$f(\tau) = \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-w(\mu^2 + 2|\tau|^\alpha \mu \cos \frac{\pi\alpha}{2} + |\tau|^{2\alpha})} dw$$

where

$$e^{-2\mu \cos \frac{\pi\alpha}{2} |\tau|^\alpha w} = \mathbb{E} e^{i\tau S_2(w)} = \widehat{g}_2(\tau, w) \quad \text{and} \quad e^{-|\tau|^{2\alpha} w} = \mathbb{E} e^{i\tau S_1(w)} = \widehat{g}_1(\tau, w).$$

Thus,

$$f(\tau) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} [\widehat{g}_1(\tau, W_\beta) \widehat{g}_2(\tau, W_\beta)]$$

from which, we immediately get that

$$\begin{aligned} \text{Cov}_X(h) &= \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\int_{-\infty}^{+\infty} g_1(h-z, W_\beta) g_2(z, W_\beta) dz \right] \\ &= \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-w\mu^2} \int_{-\infty}^{+\infty} g_1(h-z, w) g_2(z, w) dz dw \end{aligned}$$

□

3. FRACTIONAL POWERS OF HIGHER-ORDER OPERATORS

We focus our attention on the following equation

$$\left(\mu - \frac{d^2}{dt^2}\right)^\beta X(t) = \mathcal{E}(t), \quad t \geq 0, \quad \mu > 0, \quad \beta > 0 \quad (3.1)$$

that is, to the equation (1.4) for $n = 1$.

Theorem 3.1. *The representation of a solution to the equation (3.1) can be written as*

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-\mu w} \int_{-\infty}^{+\infty} u_2(x, w) \mathcal{E}(t+x) dx dw, \quad \beta > 0, \quad \mu > 0. \quad (3.2)$$

Moreover, the spectral function of (3.1) reads

$$f(\tau) = \frac{\sigma^2}{(\mu + \tau^2)^{2\beta}} \quad (3.3)$$

and the corresponding covariance function has the form

$$\text{Cov}_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\frac{e^{-\frac{h^2}{4W_{2\beta}}}}{2\sqrt{\pi W_{2\beta}}} \right] = \frac{2\sigma^2}{\Gamma(2\beta)} \left(\frac{|h|}{2\sqrt{\mu}} \right)^{2\beta} K_{2\beta}(|h|\sqrt{\mu}) \quad (3.4)$$

where $W_{2\beta}$ is a gamma r.v. with parameters $\mu, 2\beta$.

Proof. We can formally write

$$e^{w \frac{d^2}{dt^2}} = \int_{-\infty}^{\infty} e^{x \frac{d}{dt}} \frac{e^{-\frac{x^2}{4w}}}{2\sqrt{\pi w}} dx \quad (3.5)$$

so that from (3.1) we have that

$$\begin{aligned} X(t) &= \frac{1}{\Gamma(\beta)} \int_0^{\infty} e^{-\mu w} w^{\beta-1} dw \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{4w}}}{2\sqrt{\pi w}} e^{x \frac{d}{dt}} \mathcal{E}(t) dx \\ &= \frac{1}{\Gamma(\beta)} \int_0^{\infty} e^{-\mu w} w^{\beta-1} dw \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{4w}}}{2\sqrt{\pi w}} \mathcal{E}(t+x) dx. \end{aligned} \quad (3.6)$$

By observing that

$$\mathbb{E} \mathcal{E}(t+x_1) \mathcal{E}(t+h+x_2) = \begin{cases} \sigma^2, & x_1 - x_2 = h \\ 0, & \text{otherwise} \end{cases}$$

we can write

$$\begin{aligned} \mathbb{E} X(t) X(t+h) &= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^{\infty} e^{-\mu w_1} w_1^{\beta-1} dw_1 \int_0^{\infty} e^{-\mu w_2} w_2^{\beta-1} dw_2 \int_{-\infty}^{\infty} \frac{e^{-\frac{x_1^2}{4w_1}}}{2\sqrt{\pi w_1}} \frac{e^{-\frac{(h-x_1)^2}{4w_2}}}{2\sqrt{\pi w_2}} dx_1 \\ &= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^{\infty} e^{-\mu w_1} w_1^{\beta-1} dw_1 \int_0^{\infty} e^{-\mu w_2} w_2^{\beta-1} dw_2 \frac{e^{-\frac{h^2}{4(w_1+w_2)}}}{2\sqrt{\pi(w_1+w_2)}} \\ &= \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\frac{e^{-\frac{h^2}{4(W_1+W_2)}}}{2\sqrt{\pi(W_1+W_2)}} \right] \\ &= \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\frac{e^{-\frac{h^2}{4W}}}{2\sqrt{\pi W}} \right] \\ &= \frac{\sigma^2}{\Gamma(2\beta)} \int_0^{\infty} \frac{e^{-\frac{h^2}{4w}}}{2\sqrt{\pi w}} w^{2\beta-1} e^{-\mu w} dw \\ &= \frac{2\sigma^2}{\Gamma(2\beta)} \left(\frac{h}{2\sqrt{\mu}} \right)^{2\beta} K_{2\beta}(h\sqrt{\mu}) \end{aligned}$$

We notice that

$$\text{Cov}_X(h) = \frac{\sigma^2}{\mu^{2\beta}} P(B(W_{2\beta}) \in dh) / dh$$

where $B(W_{2\beta})$ is a Brownian motion with random time $W_{2\beta}$. Thus, we obtain that

$$f(\tau) = \int_{-\infty}^{\infty} e^{i\tau h} \text{Cov}_X(h) dh = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^{\infty} e^{-w\tau^2} w^{2\beta-1} e^{-\mu w} dw = \frac{\sigma^2}{(\mu + \tau^2)^{2\beta}}.$$

□

An alternative representation of the covariance function above reads

$$\begin{aligned}\mathbb{E}X(t)X(t+h) &= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty e^{-\mu w_1} w_1^{\beta-1} dw_1 \int_0^\infty e^{-\mu w_2} w_2^{\beta-1} dw_2 \int_{-\infty}^\infty \frac{e^{-\frac{x_1^2}{4w_1}}}{2\sqrt{\pi w_1}} \frac{e^{-\frac{(x_1-h)^2}{4w_2}}}{2\sqrt{\pi w_2}} dx_1 \\ &= 4\sigma^2 \int_{-\infty}^{+\infty} \left(\frac{|x_1||x_2-h|}{4\mu} \right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}(\sqrt{\mu}|x_1|) K_{\beta-\frac{1}{2}}(\sqrt{\mu}|x_1-h|) dx_1.\end{aligned}$$

We now pass to the general even-order fractional equation (1.4).

Theorem 3.2. *The representation of a solution to the equation (1.4) can be written as*

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-\mu w} \int_{-\infty}^{+\infty} u_{2n}(x, w) \mathcal{E}(t+x) dx dw, \quad \beta > 0, \mu > 0. \quad (3.7)$$

Moreover, the spectral function of (3.7) is

$$f(\tau) = \frac{\sigma^2}{(\mu + \tau^{2n})^{2\beta}} \quad (3.8)$$

and the covariance function reads

$$\text{Cov}_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E}[u_{2n}(h, W_{2\beta})] \quad (3.9)$$

where $W_{2\beta}$ is a gamma r.v. with parameters $\mu, 2\beta$.

Proof. The solution $u_{2n}(x, t)$ to

$$\frac{\partial}{\partial t} u_{2n} = (-1)^{n+1} \frac{\partial^{2n}}{\partial x^{2n}} u_{2n} \quad (3.10)$$

has Fourier transform

$$U(\beta, t) = e^{(-1)^{n+1}(-i\beta)^{2n}t} = e^{-\beta^{2n}t}. \quad (3.11)$$

We write

$$e^{-w \frac{\partial^{2n}}{\partial t^{2n}}} = \int_{-\infty}^\infty e^{ix \frac{\partial}{\partial t}} u_{2n}(x, w) dx. \quad (3.12)$$

Since

$$U(-i\beta, t) = e^{-(-1)^n \beta^{2n}t}, \quad (3.13)$$

we also write

$$e^{-w(-1)^n \frac{\partial^{2n}}{\partial t^{2n}}} = \int_{-\infty}^\infty e^{x \frac{\partial}{\partial t}} u_{2n}(x, w) dx. \quad (3.14)$$

In conclusion, we have that

$$X(t) = \left(\mu + (-1)^n \frac{\partial^{2n}}{\partial t^{2n}} \right)^{-\beta} \mathcal{E}(t) \quad (3.15)$$

$$\begin{aligned}&= \frac{1}{\Gamma(\beta)} \int_0^\infty dw e^{-\mu w} w^{\beta-1} \left(\int_{-\infty}^{+\infty} dx u_{2n}(x, w) e^{x \frac{\partial}{\partial t}} \mathcal{E}(t) \right) \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty dw e^{-\mu w} w^{\beta-1} \int_{-\infty}^{+\infty} dx u_{2n}(x, w) \mathcal{E}(t+x)\end{aligned} \quad (3.16)$$

and this confirms (3.7).

From (3.7), in view of (2.20), we obtain

$$\begin{aligned}\mathbb{E}X(t)X(t+h) &= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty dw_1 w_1^{\beta-1} e^{-\mu w_1} \int_0^\infty dw_2 w_2^{\beta-1} e^{-\mu w_2} \\ &\quad \cdot \int_{-\infty}^{+\infty} dx_1 u_{2n}(x_1, w_1) \int_{-\infty}^{+\infty} dx_2 u_{2n}(x_2, w_2) \delta(x_2 - x_1 + h)\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty dw_1 w_1^{\beta-1} e^{-\mu w_1} \int_0^\infty dw_2 w_2^{\beta-1} e^{-\mu w_2} \\
&\quad \cdot \int_{-\infty}^{+\infty} dx_1 u_{2n}(x_1, w_1) u_{2n}(x_1 - h, w_2) \\
&= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty dw_1 w_1^{\beta-1} e^{-\mu w_1} \int_0^\infty dw_2 w_2^{\beta-1} e^{-\mu w_2} u_{2n}(h, w_1 + w_2) \\
&= \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} u_{2n}(h, W_1 + W_2).
\end{aligned}$$

By following the same arguments as in the previous proof, we get that

$$\mathbb{E} X(t) X(t+h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} u_{2n}(h, W_{2\beta}) = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^\infty dw w^{2\beta-1} e^{-\mu w} u_{2n}(h, w)$$

The spectral density of $X(t)$ is therefore

$$f(\tau) = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^\infty dw w^{2\beta-1} e^{-\mu w - \tau^{2n} w} = \frac{\sigma^2}{(\mu + \tau^{2n})^{2\beta}}.$$

□

Theorem 3.2 extends the results of Theorem 3.1 when even-order heat-type equations are involved.

We now pass to the study of the equation (1.9) for $n = 1$ and $\kappa = \mp 1$,

$$\left(\mu + \kappa \frac{d^3}{dt^3} \right)^\beta X(t) = \mathcal{E}(t). \quad (3.17)$$

Theorem 3.3. *The representation of a solution to the equation (3.17) can be written as*

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\mu w} dw w^{\beta-1} \int_{-\infty}^\infty \frac{1}{\sqrt[3]{3w}} Ai\left(\frac{\kappa x}{\sqrt[3]{3w}}\right) \mathcal{E}(t+x) dx dw, \quad \beta > 0, \mu > 0. \quad (3.18)$$

Moreover,

$$f(\tau) = \frac{\sigma^2}{(\mu + \kappa i \tau^3)^{2\beta}} \quad (3.19)$$

and

$$\text{Cov}_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\frac{\sigma^2}{\sqrt[3]{3W}} Ai\left(\frac{-\kappa h}{\sqrt[3]{3W}}\right) \right] \quad (3.20)$$

where $Ai(x)$ is the Airy function and W is a gamma-distributed r.v. with parameters 2β and μ .

Proof. By following the approach adopted above, after some calculation we can write that

$$X^-(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-\mu w + w \frac{d^3}{dt^3}} \mathcal{E}(t) dw \quad (3.21)$$

is the solution to

$$\left(\mu - \frac{d^3}{dt^3} \right)^\beta X(t) = \mathcal{E}(t) \quad (3.22)$$

whereas

$$X^+(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-\mu w - w \frac{d^3}{dt^3}} \mathcal{E}(t) dw \quad (3.23)$$

is the solution to

$$\left(\mu + \frac{d^3}{dt^3} \right)^\beta X(t) = \mathcal{E}(t) \quad (3.24)$$

The third-order heat type equation

$$\frac{\partial}{\partial t} u = \kappa \frac{\partial^3}{\partial x^3} u, \quad u(x, 0) = 0, \quad (3.25)$$

has solution, for $\kappa = -1$,

$$u(x, t) = \frac{1}{\sqrt[3]{3t}} \text{Ai} \left(\frac{x}{\sqrt[3]{3t}} \right), \quad x \in \mathbb{R}, t > 0, \quad (3.26)$$

with Fourier transform

$$\int_{-\infty}^{\infty} e^{i\beta x} u(x, t) dx = e^{-it\beta^3}. \quad (3.27)$$

Formula (3.27) leads to the integral

$$\int_{-\infty}^{\infty} e^{\theta x} u(x, t) dx = e^{t\theta^3}, \quad \theta \in \mathbb{R}$$

because of the asymptotic behaviour of the Airy function (see [1] and [9]). The solution to (1.9) with $n = 1$ (that is $\kappa = -1$) is therefore (3.21).

The equation (3.25) has solution, for $\kappa = +1$, given by

$$u(x, t) = \frac{1}{\sqrt[3]{3t}} \text{Ai} \left(\frac{-x}{\sqrt[3]{3t}} \right), \quad x \in \mathbb{R}, t > 0. \quad (3.28)$$

Thus, by following the same reasoning as before, we arrive at

$$\int_{-\infty}^{\infty} e^{\theta x} u(x, t) dx = e^{-t\theta^3}, \quad \theta \in \mathbb{R}$$

and we obtain that (3.23) solves (3.17) with $\kappa = +1$ is (3.23).

In light of (2.20) we get

$$\begin{aligned} \mathbb{E}[X^-(t) X^-(t+h)] &= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty e^{-\mu w_1} dw_1 w_1^{\beta-1} \int_0^\infty e^{-\mu w_2} dw_2 w_2^{\beta-1} \\ &\quad \cdot \int_{-\infty}^\infty \frac{1}{\sqrt[3]{3w_1}} \text{Ai} \left(\frac{x_1}{\sqrt[3]{3w_1}} \right) \frac{1}{\sqrt[3]{3w_2}} \text{Ai} \left(\frac{x_1 - h}{\sqrt[3]{3w_2}} \right) dx_1 \\ &= \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty e^{-\mu w_1} dw_1 w_1^{\beta-1} \int_0^\infty e^{-\mu w_2} dw_2 w_2^{\beta-1} \\ &\quad \cdot \frac{1}{\sqrt[3]{3(w_1 + w_2)}} \text{Ai} \left(\frac{h}{\sqrt[3]{3(w_1 + w_2)}} \right) \\ &= \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\frac{1}{\sqrt[3]{3W_{2\beta}}} \text{Ai} \left(\frac{h}{\sqrt[3]{3W_{2\beta}}} \right) \right]. \end{aligned}$$

From the Fourier transform (3.27), we get that

$$\begin{aligned} f^-(\tau) &= \frac{\sigma^2}{\mu^{2\beta}} \int_{\mathbb{R}} e^{i\tau h} \mathbb{E} \left[\frac{1}{\sqrt[3]{3W_{2\beta}}} \text{Ai} \left(\frac{h}{\sqrt[3]{3W_{2\beta}}} \right) \right] dh \\ &= \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[e^{-i\tau^3 W_{2\beta}} \right] \\ &= \frac{\sigma^2}{(\mu + i\tau^3)^{2\beta}} \\ &= \frac{\sigma^2 e^{-i2\beta \arctan \frac{\tau^3}{\mu}}}{(\mu^2 + \tau^6)^\beta}. \end{aligned}$$

Also, we obtain that

$$\mathbb{E}[X^+(t) X^+(t+h)] = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\frac{1}{\sqrt[3]{3W_{2\beta}}} \text{Ai} \left(\frac{-h}{\sqrt[3]{3W_{2\beta}}} \right) \right].$$

with

$$f^+(\tau) = \frac{\sigma^2}{(\mu - i\tau^3)^{2\beta}} = \frac{\sigma^2 e^{+i2\beta \arctan \frac{\tau^3}{\mu}}}{(\mu^2 + \tau^6)^\beta}. \quad (3.29)$$

□

Theorem 3.4. *The representation of a solution to the following equation*

$$\left(\mu + \kappa \frac{\partial^{2n+1}}{\partial t^{2n+1}} \right)^\beta X(t) = \mathcal{E}(t), \quad n = 1, 2, \dots$$

reads

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-\mu w} \int_{-\infty}^{+\infty} u_{2n+1}(\kappa x, w) \mathcal{E}(t+x) dw dx, \quad \beta > 0, \mu > 0.$$

Moreover, the covariance function

$$\text{Cov}_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} u_{2n+1}(\kappa h, W_{2\beta}).$$

has Fourier transform

$$f(\tau) = \frac{\sigma^2}{(\mu + \kappa i \tau^{2n+1})^{2\beta}} = \frac{\sigma^2 e^{-i2\beta \kappa \arctan \frac{\tau^{2n+1}}{\mu}}}{(\mu^2 + \tau^{2(2n+1)})^\beta}.$$

Proof. The proof follows the same lines as in the previous theorem. □

REFERENCES

- [1] G. Accetta, E. Orsingher, Asymptotic expansion of fundamental solutions of higher order heat equations. *Random Oper. Stochastic Equations*, 5 (1997) 217–226.
- [2] J. Angulo, M. Kelbert, N. Leonenko, M.D. Ruiz-Medina, Spatiotemporal random fields associated with stochastic fractional Helmholtz and heat equations. *Stochastic Environmental Research & Risk Assessment* **22** (2008) s3-s13.
- [3] R. Gay, C. C. Heyde, On a class of random field models which allows long range dependence. *Biometrika*, 77 (1990) 401–403.
- [4] I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, Series and Products*, Accademic Press, Boston, (1994).
- [5] M.Y. Kelbert, N.N. Leonenko, M.D. Ruiz-Medina, Fractional random fields associated with stochastic fractional heat equations. *Advances in Applied Probability* 37(1), 108–133.
- [6] V. Ju. Krylov, Some properties of the distribution corresponding to the equation $\partial u / \partial t = (-1)^{p+1} \partial^{2q} u / \partial x^{2q}$. *Dokl. Akad. Nauk SSSR* 132 1254–1257 (Russian); translated as *Soviet Math. Dokl.* 1 (1960) 760–763.
- [7] A. Lachal, Distributions of sojourn time, maximum and minimum for pseudo-processes governed by higher-order heat-type equations. *Electron. J. Probab.* 8 (2003), no. 20, 1–53.
- [8] N. N. Lebedev, *Special functions and their applications*, Dover, New York (1972).
- [9] X. Li, R. Wong, Asymptotic behaviour of the fundamental solution to $\partial u / \partial t = -(-\Delta)^m u$. *Proceedings: Mathematical and Physical Sciences*, 441 (1993) 423 – 432.
- [10] E. Orsingher, M. D'Ovidio, Probabilistic representation of fundamental solutions to $\frac{\partial u}{\partial t} = \kappa_m \frac{\partial^m u}{\partial x^m}$. *Electronic Communications in Probability*, 17, (2012), 1 – 12.

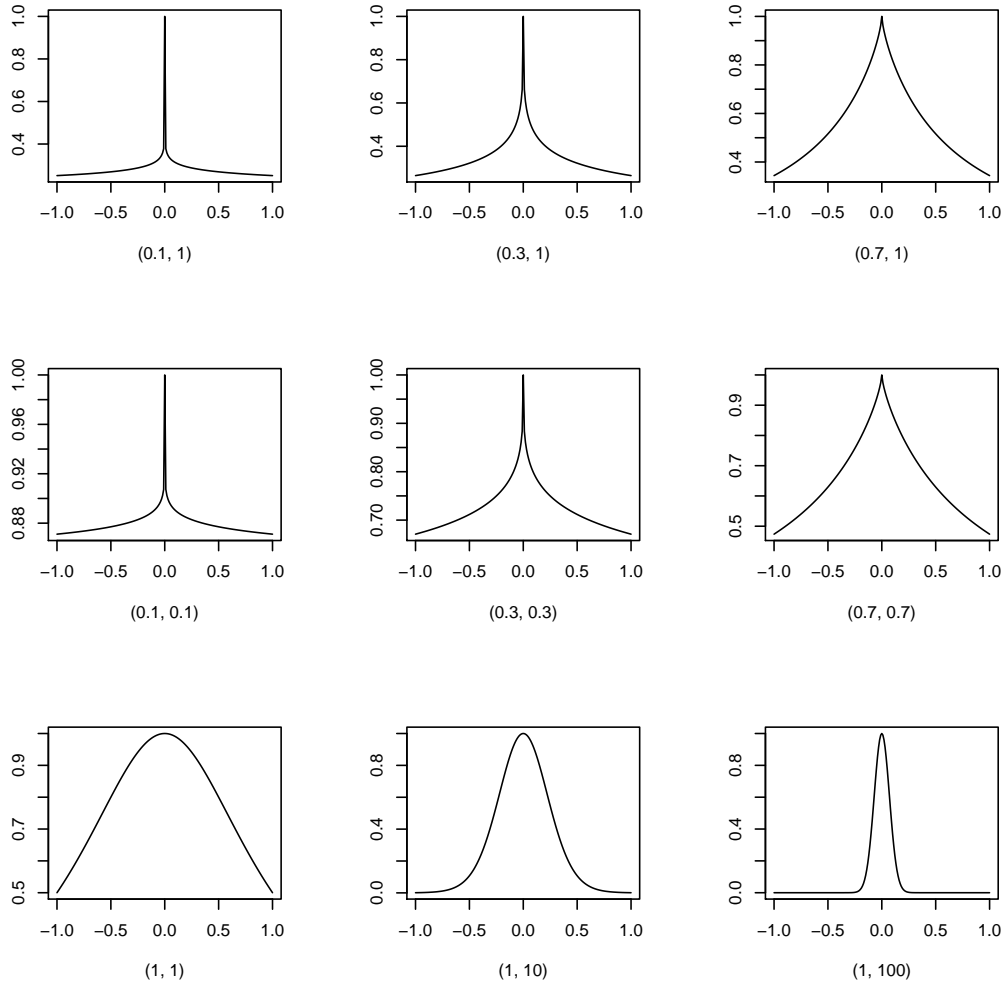


FIGURE 1. The spectral function (1.3) with different values for the parameters (α, β) .

- [11] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives. *Gordon and Breach Science Publishers*, 1993

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